Renormalization group flows and quantum phase transitions: fidelity versus entanglement

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We compare the roles of fidelity and entanglement in characterizing renormalization group flows and quantum phase transitions. It turns out that the scaling parameter extracted from fidelity for different ground states succeeds to capture nontrivial information including stable and unstable fixed points, whereas the von Neumann entropy as a bipartite entanglement measure (or equivalently, majorization relations satisfied by the spectra of the reduced density matrix along renormalization group flows) often fails, as far as the intrinsic irreversibility-information loss along renormalization group flows-is concerned. We also clarify an intimate connection between the von Neumman entropy, majorization relations, and fidelity. The relevance to Zamolodchikov's c theorem is indicated.

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Quantum phase transitions (QPTs) [1] are a topic of current interest, subject to intense study in condensed matter systems. Conventionally, a QPT is described in terms of a Hamiltonian and its spectrum. Most examples of the well-studied QPTs fit into the Landau-Ginzburg-Wilson paradigm, with the central concept being a local order parameter. The non-zero expectation value of a local order parameter characterizes a symmetry-breaking phase that only occurs for a system with infinite number of degrees of freedom, in contrast to QPTs resulting from a level crossing in a finite size system. However, there exist phases beyond symmetry-breaking orders, which yields exotic QPTs characterized by topological/quantum orders [2]. Unconventional QPTs also occur in matrix product systems [3].

Another picture emerges due to latest advances in quantum information science, which attempts to characterize QPTs from both entanglement [4, 5, 6, 7, 8] (for a review, see [9]) and fidelity [10, 11, 12] perspectives. Remarkably, for quantum spin chains, the von Neumann entropy, as a bipartite entanglement measure, exhibits qualitatively different behaviors at and off criticality [6]. On the other hand, it has been shown that fidelity may be used to characterize QPTs, regardless of what type of internal order is present in quantum many-body states [11]. A basic point is that there always exists an order parameter (local/nonlocal) in a system undergoing a QPT. In principle, such an order parameter may be constructed systematically [13]. This results in the orthogonality of different ground states, due to state distinguishability [14]. Thus, it is justified that the scaling parameter may be extracted from fidelity to characterize QPTs. In fact, nontrivial information about QPTs including stable and unstable fixed points along renormalization group (RG) flows may be revealed solely from ground states, and critical exponents can be extracted by performing a finite size scaling analysis from such a scaling parameter [15].

An intriguing question is to clarify the connection between fidelity and entanglement approaches to QPTs. An apparent difference between entanglement measures and fidelity is that entanglement measures involve *partitions*, whereas the fidelity approach treats systems as a *whole*. Therefore, *no matter what bipartite entanglement measures are used, some physical information will be lost, since the whole is not simply the sum of the parts.* On the other hand, one may expect that the scaling parameter extracted from fidelity should capture information that is lost in bipartite entanglement measures.

In this paper, we compare the roles of fidelity and entanglement in characterizing RG flows and QPTs. It is found that the scaling parameter extracted from fidelity for different ground states succeeds to capture nontrivial information including stable and unstable fixed points, whereas the von Neumann entropy as a bipartite entanglement measure (or equivalently, majorization relations satisfied by the spectra of the reduced density matrix along RG flows) often fails, as far as the intrinsic irreversibility, i.e., information loss along RG flows is concerned. We also clarify an intimate connection between the von Neumman entropy, majorization relations and fidelity. The relevance to Zamolodchikov's c theorem is indicated, if a system is conformally invariant at transition points,.

Generalities. For a quantum spin chain described by a Hamiltonian $H(\lambda)$, with λ a control parameter, it is well-established [16] that its ground state $|\psi(\lambda)\rangle$ may be represented in terms of the so-called matrix product states (MPS) [17]. Suppose the system is translationally invariant, then $|\psi(\lambda)\rangle$ takes the form,

$$|\psi(\lambda)\rangle = \mathrm{Tr}(A_{i_1}(\lambda)\cdots A_{i_L}(\lambda))|i_1\cdots i_L\rangle, \qquad (1)$$

where $\{A_i\}$ is a set of $dD\times D$ matrices, with d being the dimension of the local Hilbert space at each lattice site, and D the dimension of the bonds in the valence bond picture [3, 16]. The fidelity $F(\lambda,\lambda')\equiv|\langle\psi(\lambda')|\psi(\lambda)\rangle|$ for

two ground states $|\psi(\lambda)\rangle$ and $|\psi(\lambda')\rangle$ corresponding to different values of the control parameter λ is

$$F(\lambda, \lambda') = \mathrm{Tr}(E^L(\lambda, \lambda')), \tag{2}$$

with the (generalized) transfer matrix $E(\lambda, \lambda') = \sum_{s=1}^{d} A^{s*}(\lambda') \otimes A^s(\lambda)$. Therefore, it is straightforward to evaluate the fidelity F by diagonalizing the transfer matrix $E(\lambda, \lambda')$. For large enough L's, the fidelity F scales as $F(\lambda, \lambda') \sim d^L(\lambda, \lambda')$, where $d(\lambda, \lambda')$ is the scaling parameter. In the thermodynamic limit, $d(\lambda, \lambda')$ may be defined as

$$\ln d(\lambda, \lambda') = \lim_{L \to \infty} \ln F(\lambda, \lambda')/L. \tag{3}$$

In fact, the scaling parameter $d(\lambda, \lambda)$ is the largest eigenvalue of the transfer matrix $E(\lambda, \lambda')$, if there is no pair of eigenvalues that are complex conjugate each other (otherwise, a fast oscillating part should first be factored out [11]). Obviously, it satisfies the properties inherited from the fidelity $F(\lambda, \lambda')$: (a) $d(\lambda, \lambda) = 1$; (b) $d(\lambda, \lambda') = d(\lambda', \lambda)$; and (c) $0 \leq d(\lambda, \lambda') \leq 1$.

Although the system is in a pure state, a subsystem A consisting of N consecutive spins is generically in a mixed state described by the reduced density matrix $\rho_A(\lambda) = \mathrm{Tr}_B \rho(\lambda)$, with the density matrix $\rho(\lambda) = |\psi(\lambda)\rangle\langle\psi(\lambda)|$, and the trace is taken over the subsystem B complementary to the subsystem A. The fidelity $F_A(\lambda, \lambda')$ for two reduced density matrices $\rho_A(\lambda)$ and $\rho_A(\lambda')$ is defined as $F_A(\lambda, \lambda') = \mathrm{Tr}\sqrt{\rho_A^{1/2}(\lambda)\rho_A(\lambda')\rho_A^{1/2}(\lambda)}$. For large enough L's and N's, with a fixed N/L [18], F_A scales as $F_A(\lambda, \lambda') \sim d_A^N(\lambda, \lambda')$, where $d_A(\lambda, \lambda')$ is the scaling parameter for the subsystem A. Formally, in the thermodynamic limit, $d_A(\lambda, \lambda')$ may be defined as

$$\ln d_A(\lambda, \lambda') = \lim_{N \to \infty} \ln F_A(\lambda, \lambda')/N. \tag{4}$$

The scaling parameter $d_A(\lambda, \lambda)$ also enjoys the properties: (a) $d_A(\lambda, \lambda) = 1$; (b) $d_A(\lambda, \lambda') = d_A(\lambda', \lambda)$; and (c) $0 \leq d_A(\lambda, \lambda') \leq 1$. The scaling parameters $d(\lambda, \lambda')$ and $d_A(\lambda, \lambda')$ satisfy

$$d_A(\lambda, \lambda') \geq d(\lambda, \lambda'). \tag{5}$$

This follows from the well-known inequality: $F_A(\lambda, \lambda') \geq F(\lambda, \lambda')$ [14]. Although it is straightforward to calculate $d(\lambda, \lambda')$ in the MPS representation, it is a formidable task to evaluate $d_A(\lambda, \lambda')$. This is due to the fact that $d_A(\lambda, \lambda')$ depends not only on the spectra but also on the eigenvectors of the reduced density matrices $\rho_A(\lambda)$ and $\rho_A(\lambda')$. From the Schmidt decomposition, one may expect that $d_A(\lambda, \lambda')$ takes the form $d_A(\lambda, \lambda') = \sum_{\alpha,\beta} g_{\alpha\beta}(\lambda, \lambda')\sqrt{\omega_\alpha(\lambda)}\sqrt{\omega_\beta(\lambda')}$, with $\alpha, \beta = 1, \cdots, D$ and $\{w_\alpha(\lambda)\}$ the (normalized and non-increasing ordered) spectra of the reduced density matrix ρ_A, and $g_{\alpha\beta}(\lambda, \lambda')$ a model-dependent real tensor, satisfying $g_{\alpha\beta}(\lambda, \lambda') = g_{\beta,\alpha}(\lambda', \lambda)$. The difficulty arises from the dependence of $g_{\alpha\beta}(\lambda, \lambda')$ on λ and λ'. Thus, instead of $d_A(\lambda, \lambda')$, we may seek a quantity $d_E(\lambda, \lambda') = \sum_{\alpha,\beta} E_{\alpha\beta}\sqrt{\omega_\alpha(\lambda)}\sqrt{\omega_\beta(\lambda')}$, which is universal in the sense that the real tensor $E_{\alpha\beta}$ is independent of λ and λ'. The only choice is $E_{\alpha\beta} = \delta_{\alpha\beta}$, if one requires that $d_E(\lambda, \lambda')$ enjoys the same properties as $d(\lambda, \lambda')$ and $d_A(\lambda, \lambda')$: (a) $d_E(\lambda, \lambda) = 1$; (b) $d_E(\lambda, \lambda') = d_E(\lambda', \lambda)$; and (c) $0 \leq d_E(\lambda, \lambda') \leq 1$. Thus $d_E(\lambda, \lambda') = \sum_\alpha \sqrt{\omega_\alpha(\lambda)}\sqrt{\omega_\alpha(\lambda')}$. We stress that $d_E(\lambda, \lambda')$ only involves the eigenvalues of the reduced density matrix, whereas $d(\lambda, \lambda')$ unveils information encoded in the entire ground state.

The connection between $d(\lambda, \lambda')$, $d_E(\lambda, \lambda')$ and majorization. Suppose a system flows to two different stable fixed points λ_- and λ_+ under RG transformations [19], with an unstable fixed point λ_c as a transition point. An intriguing fact is that $d_E(\lambda, \lambda')$ shares similar behaviors to $d(\lambda, \lambda')$, if λ and λ' are in the same phase. To demonstrate this, we notice that, generically, there is one and only one extreme point for $d_E(\lambda, \lambda')$ if one regards $d_E(\lambda, \lambda')$ as a function of λ for a given λ'. That is, when $\lambda = \lambda'$, it reaches the maximum 1, as follows from the Lagrange multipliers for $d_E(\lambda, \lambda')$ under the constraints: $\sum_\alpha \omega_\alpha(\lambda) = 1$ and $\sum_\alpha \omega_\alpha(\lambda') = 1$. Therefore, $d_E(\lambda, \lambda')$ increases with λ, until it reaches 1 when $\lambda = \lambda'$, then it decreases with λ along an RG flow for a fixed λ'.

We stress that the monotonic behaviors of $d_E(\lambda, \lambda')$ are consistent with majorization relations for the spectra of the reduced density matrix along RG flows. It has been established that there is a more "fine-grained" characterization of entanglement loss along RG flows in terms of majorization [20, 21, 22], which states that $m_k \equiv \sum_{\alpha=1}^{k} \omega_\alpha (k = 1, 2, \cdots)$ is non-decreasing along RG flows. As shown in Ref. [22], this simply follows from the fact that there is one and only one crossing point α^* for eigenvalue distributions when one regards ω_α as a function of the index α [23]. Equivalently, $\triangledown_\lambda \omega_\alpha \geq 0$ for $\alpha \leq \alpha^*$, and $\triangledown_\lambda \omega_\alpha \leq 0$ for $\alpha > \alpha^*$, where $\triangledown_\lambda$ denotes the derivative with respect to λ along RG flows. Then, suppose λ and λ' are in the same phase and λ' is fixed, $\triangledown_\lambda d_E(\lambda, \lambda') = 1/2(\sum_{\alpha \leq \alpha^*} \sqrt{\omega_\alpha(\lambda')/\omega_\alpha(\lambda)}\triangledown_\lambda \omega_\alpha(\lambda) + \sum_{\alpha > \alpha^*} \sqrt{\omega_\alpha(\lambda')/\omega_\alpha(\lambda)}\triangledown_\lambda \omega_\alpha(\lambda)) \geq 1/2\triangledown_\lambda(\sum_\alpha \omega_\alpha(\lambda)) = 0$, if λ flows from λ_c to λ', because $\omega_\alpha(\lambda')/\omega_\alpha(\lambda) \geq 1$ for $\alpha \leq \alpha^*$, and $\omega_\alpha(\lambda')/\omega_\alpha(\lambda) \leq 1$ for $\alpha > \alpha^*$; $\triangledown_\lambda d_E(\lambda, \lambda') = 1/2(\sum_{\alpha \leq \alpha^*} \sqrt{\omega_\alpha(\lambda')/\omega_\alpha(\lambda)}\triangledown_\lambda \omega_\alpha(\lambda) + \sum_{\alpha > \alpha^*} \sqrt{\omega_\alpha(\lambda')/\omega_\alpha(\lambda)}\triangledown_\lambda \omega_\alpha(\lambda)) \leq 0$, if λ flows from λ' to a stable fixed point, because $\omega_\alpha(\lambda')/\omega_\alpha(\lambda) \leq 1$ for $\alpha \leq \alpha^*$, and $\omega_\alpha(\lambda')/\omega_\alpha(\lambda) \geq 1$ for $\alpha > \alpha^*$.

If λ and λ' are in different phases, then the monotonic behaviors of $d_E(\lambda, \lambda')$ with λ along an RG flow for a fixed λ' depends on what type of orders present in systems. For our purpose, we discuss the two extreme cases: $D = 2$ for MPS systems, and $D = \infty$ for spontaneous symmetry-breaking orders. Since the monotonicity is universal in the sense that it should re-

main the same for λ''s from the same phase, therefore we just need to focus on the stable fixed points $\lambda' = \lambda_\pm$. For $D = 2$, there are only two different choices for the reduced density matrix spectra at the stable fixed points: $\{1, 0\}$ and $\{1/2, 1/2\}$, corresponding to unentangled states and maximally entangled states, respectively. For the former, $\triangledown_\lambda d_E(\lambda, \lambda') = \triangledown_\lambda \sqrt{\omega_1(\lambda)} \geq 0$, since $\omega_1(\lambda)$ is non-decreasing along an RG flow. For the latter, $\triangledown_\lambda d_E(\lambda, \lambda') = 1/\sqrt{2}\triangledown_\lambda(\sqrt{\omega_1(\lambda)} + \sqrt{\omega_2(\lambda)}) = 1/(2\sqrt{2})(1/\sqrt{\omega_1(\lambda)} - 1/\sqrt{\omega_2(\lambda)})\triangledown_\lambda \omega_1 \leq 0$. For $D = \infty$ as occurs for spontaneous symmetry-breaking orders, a remarkable feature is that the reduced density matrix spectra at critical points tend to vanish, leading to the divergence of the von Neumann entropy for a half-infinite chain. Thus, $D_E(\lambda, \lambda')$ tends to 0 when λ or λ' tends to λ_c. On the other hand, degeneracies occur in the largest eigenvalue of the reduced density matrix in symmetry-broken phases, and no degeneracy in symmetric phase. Therefore, the reduced density matrix spectra are $\{1/n, \cdots, 1/n, 0, \cdots\}$ (n-degeneracies) for stable fixed points in symmetry-broken phases and $\{1, 0, \cdots\}$ for stable fixed points in symmetric phases. Then, $d_E(\lambda, \lambda')$ as a function of λ for a fixed λ' monotonically increases, if λ' is in symmetric phases, whereas $d_E(\lambda, \lambda')$ increases, until it reaches a maximum, then decreases, if λ' is in symmetry-broken phases.

The fact that $d_E(\lambda, \lambda')$ shares similar monotonic behaviors to $d(\lambda, \lambda')$ establishes a connection between entanglement and fidelity approaches to QPTs, since information unveiled in d_E results from the reduced density matrix spectra that in turn determine the von Neumann entropy $E(\lambda) = -\sum_\alpha \omega_\alpha(\lambda) \ln \omega_\alpha(\lambda)$. The latter is non-increasing along RG flows. If a system is conformally invariant at transition points, then one may formulate an entropic version of Zamolodchikov's c theorem [24]. Therefore, the monotonicity of $d(\lambda, \lambda')$ and $d_E(\lambda, \lambda')$ is reminiscent of Zamolodchikov's c theorem.

Quantum XY spin 1/2 chain. The quantum XY spin chain is described by the Hamiltonian

$$H = -\sum_{j=-M}^{M} (\frac{1+\gamma}{2}\sigma_j^x \sigma_{j+1}^x + \frac{1-\gamma}{2}\sigma_j^y \sigma_{j+1}^y + \lambda \sigma_j^z). \quad (6)$$

Here σ_j^x, σ_j^y, and σ_j^z are the Pauli matrices at the j-th lattice site. The parameter γ denotes an anisotropy in the nearest-neighbor spin-spin interaction, whereas λ is an external magnetic field. The Hamiltonian (6) may be diagonalized exactly [25]. In the thermodynamic limit, $d(\lambda, \lambda'; \gamma)$ takes the form:

$$d(\lambda, \lambda'; \gamma) = \exp[\frac{1}{2\pi}\int_0^\pi d\alpha \ln \mathcal{F}(\lambda, \lambda'; \gamma; \alpha)], \quad (7)$$

where $\mathcal{F}(\lambda, \lambda'; \gamma; \alpha) = \cos[\vartheta(\lambda; \gamma; \alpha) - \vartheta(\lambda'; \gamma; \alpha)]/2$, with $\cos\vartheta(\lambda; \gamma; \alpha) = (\cos\alpha - \lambda)/\sqrt{(\cos\alpha - \lambda)^2 + \gamma^2 \sin^2\alpha}$. It was shown [11] that $d(\lambda, \lambda'; \gamma)$ exhibits a pinch point at $(1, 1)$, i.e., an intersection of two singular lines $\lambda = 1$ and $\lambda' = 1$. For quantum Ising model in a transverse field ($\gamma = 1$), all states for $\lambda > 1$ flow to the product state with all spins aligning in the z direction ($\lambda = \infty$), and all states for $\lambda < 1$ flow to the cat state $(|\leftarrow \cdots \leftarrow\rangle + |\rightarrow \cdots \rightarrow\rangle)/\sqrt{2}$ ($\lambda = 0$). We stress that $d(\lambda, \lambda'; \gamma)$ detects the duality between two phases: $\lambda > 1$ and $\lambda < 1$, because $d(\lambda, \lambda'; 1) = d(1/\lambda, 1/\lambda'; 1)$ [15].

The exact spectra of the reduced density matrix for a half-infinite chain take the form [26]:

$$\omega_{\{n_0, n_1, \cdots, n_\infty\}}(\lambda; \gamma) = \frac{e^{-\sum_{j=0}^{\infty} \epsilon_j n_j}}{Z(\lambda; \gamma)}, \quad (8)$$

where $\epsilon_j = (2j + 1)\epsilon$ if $\lambda > 1$, and $\epsilon_j = 2j\epsilon$ if $\lambda < 1$, and $Z(\lambda; \gamma) = (16q/(k^2 k'^2))^{1/24}$ if $\lambda > 1$, and $Z(\lambda; \gamma) = 2(16k^2/(qk'))^{1/12}$ if $\lambda < 1$, with $q \equiv \exp(-\epsilon)$. Here $\epsilon = \pi K(k')/K(k)$, with $K(k)$ being the complete elliptic integral of the first kind, k the modulus and $k' = \sqrt{1 - k^2}$. The relation between the modulus k and the anisotropy γ and the transverse field strength λ is $k = \gamma/\sqrt{\lambda^2 + \gamma^2 - 1}$ if $\lambda > 1$ and $k = \sqrt{\lambda^2 + \gamma^2 - 1}/\gamma$ if $\lambda < 1$. Therefore, if λ and λ' are in the same phase, we may derive a closed expression for $d_E(\lambda, \lambda'; \gamma)$:

$$d_E(\lambda, \lambda'; \gamma) = \frac{Z(\lambda, \lambda'; \gamma)}{\sqrt{Z(\lambda; \gamma) Z(\lambda'; \gamma)}}, \quad (9)$$

where $Z(\lambda, \lambda'; \gamma)$ takes the same form as $Z(\lambda; \gamma)$, i.e., $Z(\lambda, \lambda'; \gamma) = (16\tilde{q}/(\tilde{k}^2 \tilde{k}'^2))^{1/24}$ if $\lambda, \lambda' > 1$, and $Z(\lambda, \lambda'; \gamma) = 2(16\tilde{k}^2/(\tilde{q}\tilde{k}'))^{1/12}$ if $\lambda, \lambda' < 1$, with $\tilde{q} = \sqrt{q(\lambda) q(\lambda')}$, and $\tilde{k}$ determined from $\tilde{q} = \exp(-\tilde{\epsilon})$, with $\tilde{\epsilon} = \pi K(\tilde{k}')/K(\tilde{k})$. However, if λ and λ' are in different phases, $d_E(\lambda, \lambda'; \gamma)$ is only available in terms of an infinite sum.

The monotonic behaviors of $d_E(\lambda, \lambda'; \gamma)$ as an example for $D = \infty$ with Z_2 broken symmetry are numerically confirmed. If λ and λ' are in the same phase, $d_E(\lambda, \lambda'; \gamma)$ takes values very close to $d(\lambda, \lambda'; \gamma)$ when λ and λ' are away from the critical point. However, in contrast to $d(\lambda, \lambda'; \gamma)$, $d_E(\lambda, \lambda'; \gamma)$ fails to detect the duality between two phases for quantum Ising model in a transverse field ($\gamma = 1$). This is due to the fact that the dual unitary transformation connecting the two phases involves non-local operations, which do not affect the fidelity (and so $d(\lambda, \lambda'; \gamma)$), but do change the spectra of the reduced density matrix (and so $d_E(\lambda, \lambda'; \gamma)$). We note that the von Neumann entropy $E(\lambda)$ is decreasing along the RG flows, so it succeeds to detect the stable fixed points: $\lambda = 0$ and $\lambda = \infty$, but it fails to detect the duality.

Quantum spin 1/2 chain with three-body interactions. The system is described by a Hamiltonian with three-body interactions [27]

$$H = \sum_i 2(g^2 - 1)\sigma_i^z \sigma_{i+1}^z - (1+g)^2 \sigma_i^x + (g-1)^2 \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z. \quad (10)$$

The peculiarity of the model is that it is not conformally invariant at the transition point $g_c = 0$: in the thermodynamic limit, the correlation length diverges, and energy gap vanishes, but the ground state energy remains smooth [3]. As emphasized in Ref. [11], the parameter space should be compactified by identifying $g = +\infty$ with $g = -\infty$, due to the fact that $H(+\infty) = H(-\infty)$. Since ground states are an MPS with $A_1 = (I - \sigma^z)/2 + \sigma^-$ and $A_2 = (I + \sigma^z)/2 + g\sigma^+$ [3], one may extract the scaling parameter d as $d(g, g') = \sqrt{1 + |gg'|}/\sqrt{(1 + |g|)(1 + |g'|)}$ if g and g' are in different phases, and $d(g, g') = (1 + \sqrt{|gg'|})/\sqrt{(1 + |g|)(1 + |g'|)}$ if g and g' are in the same phase [11]. From this we read off that there are two transition points: $g = 0$ and ∞. All states for positive g flow to the product state ($g = 1$) with all spins aligning in the x direction, and all states for negative g flow to the cluster state [28] ($g = -1$).

The reduced density matrix spectra for a half-infinite chain are identical to those of ρ, which may be calculate exactly by exploiting the freedom in the set $\{A_i\}$ in order to fix the gauge: $\sum_i A_i A_i^\dagger = I$ and $\sum_i A_i^\dagger \rho A_i = \rho$ [3]. For $g > 0$, the eigenvalues of the reduced density matrix are $1/2 \pm \sqrt{g}/(1 + g)$, whereas for $g < 0$, both of them are $1/2$. Obviously, for positive g, the majorization relations are satisfied, since the larger eigenvalue $1/2 + \sqrt{g}/(1 + g)$ is monotonically increasing when g varies from 0 to 1, or varies from ∞ to 1, consistent with the fact that both $g = 0$ and $g = \infty$ are unstable fixed points, and $g = 1$ is a stable fixed point. However, the trivial spectra for negative g fail to identify another stable fixed point $g = -1$, in contrast to $d(g, g')$. One may check that $d_E(g, g') = d(g, g')$ if $g, g' > 0$, and $d_E(g, g') = 1$, i.e., $d_E(g.g')$ serves as a trivial upper bound, if $g, g' < 0$. On the other hand, $d_E(g, g') = 1/\sqrt{1 + g}$ if $0 < g < 1$ and $g' < 0$, and $d_E(g, g') = \sqrt{g/(1 + g)}$ if $1 < g < \infty$ and $g' < 0$. In this case, $d_E(g, g')$ is decreasing with g along an RG flow for a fixed g', consistent with the majorization relations. The von Neumann entropy $E(g < 0) = \ln 2$ also fails to identify the stable fixed point $g = -1$.

Quantum spin 1 chain with two-body interactions. The model is a deformation of the celebrated AKLT model [17], with the Hamiltonian

$$H = \sum_i (2 + g^2)\mathbf{S}_i\mathbf{S}_{i+1} + 2(\mathbf{S_i}\mathbf{S}_{i+1})^2 + 2(4 - g^2)(S^z)^2 + (g + 2)^2(S_i^z S_{i+1}^z)^2 + g(g + 2)\{S_i^z S_{i+1}^z, \mathbf{S}_i\mathbf{S}_{i+1}\}_+. \tag{11}$$

Note that the AKLT model corresponds to $g = -2$. The model is very similar to the previous one, although the order parameter is non-local. Indeed, the parameter space should be compactified by identifying $g = +\infty$ with $g = -\infty$, since $H(+\infty) = H(-\infty)$. The ground states are an MPS with $A_1 = -\sigma^z, A_2 = \sigma^-$ and $A_3 = g\sigma^+$ [3], thus $d(g, g') = \sqrt{1 + |gg'|}/\sqrt{(1 + |g|)(1 + |g'|)}$ if g and g' are in different phases, and $d(g, g') = (1 + \sqrt{|gg'|})/\sqrt{(1 + |g|)(1 + |g'|)}$ if g and g' are in the same phase. There are two transition points: $g = 0$ and ∞. All states for positive g flow to the state with $g = 1$, and all states for negative g flow to the state with $g = -1$.

The reduced density matrix spectra are trivial, since $\rho = 1/2\ I$ for both positive and negative g. Therefore, no information is unveiled from the spectra (and so the von Neumann entropy $E(g)$), as far as the stable fixed points are concerned. Actually, $d_E(g, g') = 1$, and $E(g) = \ln 2$. The fact that both d and d_E remain the same under interchange $g \leftrightarrow -g$ reflects that the ground states for $\pm g$ are equivalent up to local unitaries [3].

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